Introduction to numerical models used in the analysis

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1 FEM Navier-Stokes equations

In this work unsteady non-creeping flow in a "wedge" and in a "duct" is considered. This flow satisfies the Navier-Stokes equations, which in non-dimensional form are given by

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{Re} \nabla^2 \mathbf{u} - \nabla p \text{ in } \Omega \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \quad (2)$$

with initial and boundary conditions specified.

For the "wedge", the velocity, length, time and pressure are scaled by U_{bot} (the dimensional lid velocity), ℓ (half the dimensional lid extent), ℓ/U_{bot}, and ρU_{bot}^2 respectively. The Reynolds number is $Re = \rho U_{\mathrm{bot}} \ell/\mu$, where ρ and μ are density and dynamic viscosity, respectively. For the duct, the velocity, length, time and pressure are scaled by U_{in} (the peak dimensional inlet velocity), $2D$ (twice the dimensional duct height), $2D/U_{\mathrm{in}}$, and ρU_{in}^2 respectively. The Reynolds number is $Re = \rho U_{\mathrm{in}} 2D/\mu$, where ρ and μ are density and dynamic viscosity, respectively.

By looking at sub-components of the Navier-Stokes equations, the Poisson equation and the steady and unsteady Stokes equations can be written. The Poisson equation, which includes the diffusion term, is written in the form of

$$\begin{aligned} -\nabla^2 \mathbf{u} &= f \text{ in } \Omega \\ \mathbf{u} &= 0 \text{ on } \partial\Omega \end{aligned} \quad (3)$$

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where f is the forcing term. After adding the pressure force and the continuity equation the steady Stokes flow problem can be given as

$$\begin{aligned} -\nabla^2\mathbf{u} + \nabla\mathbf{p} &= f \text{ in } \Omega && (4)\\ \nabla\mathbf{u} &= 0 \text{ in } \Omega && (5)\\ \mathbf{u} &= 0 \text{ on } \partial\Omega \end{aligned}$$

here the pressure force includes the dynamic viscosity (i.e., $\mathbf{p} = \frac{p}{\mu}$). The unsteady Stokes problem includes an unsteady time dependent term

$$\begin{aligned} \frac{\partial\mathbf{u}}{\partial t} &= \nabla^2 u - \nabla\mathbf{p} + f \text{ in } \Omega && (6)\\ \nabla\mathbf{u} &= 0 \text{ in } \Omega\\ \mathbf{u} &= 0 \text{ on } \partial\Omega \end{aligned}$$

Once the convection term $\mathbf{u}\cdot\nabla\mathbf{u}$ is added, we arrive to the Navier-Stokes equations given in Equations 1.1 and 1.2 (without the boundary conditions). The convection term is non-linear (from $u \cdot u$), non-symmetric, and anisotropic. It should be also noted that the pressure in this formulation is only determined up to a constant $(u, p + C)$. To obtain a unique pressure solution, an additional condition should be included, such as

$$\int_\Omega p\, dx = 0 \qquad (7)$$

The system matrix and system forcing vector in the block form is given as

$$\begin{bmatrix} \frac{1}{Re}A + \frac{1}{\Delta t}M & 0 & -D_1^T \\ 0 & \frac{1}{Re}A + \frac{1}{\Delta t}M & -D_2^T \\ -D_1 & -D_2 & 0 \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ p^{n+1} \end{bmatrix} = \frac{1}{\Delta t}\begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1^n \\ u_2^n \\ 0 \end{bmatrix} - \frac{1}{12}(23C^n - 16C^{n-1} + 5C^{n-2}) \qquad (8)$$

where the convection term components, $C^q (q = n,\ n-1,\ n-2)$, are obtained from direct-stiffness summation of

$$\sum_{\delta=1}^{6} \left(\hat{C}_{\alpha\delta}^k\right)^q u_{g_\delta^k}^q \qquad (9)$$

over k and α.

The elemental and global matrix constructions and the block formation processes are described in the appropriate sections.

Historically one of the most important group of elements for approximating the Navier-Stokes equations is the so-called "Taylor-Hood" group of elements and they remain widely used. One of the first successful finite element approximation was calculated by Hood and Taylor [1] using continuous, piecewise quadratic velocities and continuous, piecewise linear pressures on a grid of triangular elements. Currently the name "Taylor-Hood" element is used to describe any finite element approximation using the same grid for pressure and velocity, which provide continuous approximations for velocity and pressure based on Lagrange type interpolation, where the order of approximation is one less for pressure than for the components of velocity. Thus, for example, for three-dimensional flow this would include:

- quadratic velocity and cubic pressure on a grid of tetrahedral elements.
- tricubic velocities and triquadratic pressure on a grid of hexagonal elements.

In practice, the only elements that are widely used are based on quadratic velocities and linear pressures on tetrahedra and triquadratic velocities and trilinear pressures on hexahedra and it is the latter that are most often used for three dimensional flow. In the present analysis the Stokes and Navier-Stokes equations for two dimensional channel flow are solved by Taylor-Hood $\mathbb{P}_2$ — $\mathbb{P}_1$ velocity–(continuous) pressure formulation, with triangular elements, using quadratic velocity and linear pressure nodes.

2 Non-linear elastic deformation

This section will be added later.

3 Coupled problems

This section will be added later.

[1] C. Taylor and P. Hood, *A numerical solution of the Navier-Stokes equations using the finite element technique*, Computers and Fluid, **1**, 73-100 (1973)